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The entropy of an irreversible quantum dynamics

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Abstract. We consider an irreversible quantum dynamical system that mimics the classical phase doubling map $z \mapsto z^2$ on the unit circle and study its ergodic properties. The main result of the paper is the computation of the dynamical entropy (log 2) using compact perturbations of unity as operational partitions of unity.

1. Introduction

A standard example of an irreversible classical dynamical system is the map

$$\theta : [0,1[\to [0,1[: x \mapsto 2x \bmod 1$$

which leaves the Lebesgue measure invariant. Another version of this system is the phase doubling map $z \mapsto z^2$ with a Lebesgue measure on the unit circle. Yet another equivalent formulation is the left-shift of the spin chain $\{0,1\}^{\mathbb{N}}$ with the Bernoulli measure that assigns a probability of 1/2 to both 0 and 1. In this paper, we will stay with the first picture.

The phase doubling map is chaotic, meaning that there exists a positive Lyapunov exponent λ, defined as

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \log |D_x(\theta^n(x))| = \log 2.$$

With quantization in mind, we can use the shift

$$T_{x_0} : [0,1[\to [0,1[: x \mapsto x + x_0 \bmod 1$$

to put the notion of the Lyapunov exponent in a form suitable for an algebraic description by writing

$$\theta \circ T_{x_0} = T_{2x_0} \circ \theta. \tag{1}$$

This expresses the simple fact that shifting a point over a distance x_0 and applying the dynamics produces the same effect as first applying the dynamics and then shifting over twice the original distance.

It is our aim in this paper to introduce an irreversible dynamic Θ on $\mathcal{B}(\mathcal{L}^2([0,1], \mathrm{d}x))$ which corresponds to the classical phase doubling and to study its entropy. We will consider

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for Θ a non-invertible, unity preserving, $*$-homomorphism and impose that it acts on the multiplication operators in the same way as the classical dynamics

$$\Theta(M_f) = M_{f\circ\theta} \tag{2}$$

M_f being the multiplication operator on $\mathcal{H} = \mathcal{L}^2([0,1], \mathrm{d}x)$ by the function $f \in \mathcal{L}^\infty([0,1], \mathrm{d}x)$. Let us stress, that we consider Θ as a dynamic on the whole of $\mathcal{B}(\mathcal{H})$, not only as a map on the multiplication operators. This approach is quite different from the approach taken in, for example, Baker [1], where a unitary evolution on a finite-dimensional vector space is constructed to mimic the classical Baker transformation. There is certainly not a unique homomorphism satisfying (2). Therefore, we impose some additional conditions on the dynamics.

Consider the unitary operator

$$(U_{x_0}\varphi)(x) := \varphi(T_{-x_0}x) \qquad \varphi \in \mathcal{H}$$

and the automorphism $A \mapsto \tau_{x_0}(A) := U_{x_0} A U_{x_0}^*$ it implements. τ_{x_0} corresponds on the level of the observables to a right-shift over a distance x_0 in position space. It is natural to ask that for any $x_0 \in [0, 1[$

$$\tau_{x_0} \circ \Theta = \Theta \circ \tau_{2x_0} \tag{3}$$

since this relation, restricted to multiplication operators, yields (1) again. Following the set-up of [2], expression (3) indicates the existence of a quantum Lyapunov exponent $\log 2$. One could say that the dynamics stretches the position observable by a factor 2. As Θ conserves the commutation relation between position and momentum it is to be expected that the momentum observable will shrink by a factor 2 under the dynamics. To see this, we introduce the group of automorphisms $\{\sigma_k \mid k \in \mathbb{Z}\}$ determined by the unitaries M_{ψ^k}, where $\psi^k(x) := \exp(\mathrm{i}2\pi kx)$. σ_k describes on the level of the observables a shift in momentum space. It follows then immediately from (2) that

$$\sigma_{2k} \circ \Theta = \Theta \circ \sigma_k \tag{4}$$

showing the presence of a second Lyapunov exponent $-\log 2$.

In section 3, we deal with the problem of defining Θ in detail. In section 4, we will discuss the ergodic properties of the dynamics: Θ has a unique invariant state and is mixing.

The basic ingredient for the construction of the Kolmogorov–Sinai invariant of a classical dynamical system is a partition of the phase space of the system into disjoint sets. The dynamics induces an asymptotic refinement on such partitions and this is measured in terms of an entropy. A similar scheme can be used for quantum dynamical systems by replacing partitions of the phase space by 'operational partitions of unity'. The resulting construction, which is based on an idea of Lindblad, has been presented in [3]. We will briefly sketch it in the preliminaries section.

However, whereas in the classical case any finite partition of phase space can be used as a starting point, one has to be much more careful in the non-commutative situation. Along with the dynamics, the non-commutativity of the system is a source of entropy and one should only allow 'reasonable' partitions. In fact, asking for the proper partitions of unity is a mathematical matter which can be expressed in physical terms by considering the question of which measurements are physically allowed. Indeed, it is by means of a partition of unity that one creates a coupling between the physical system and a measuring device (an array of spin-s objects). For example, measurements that produce by themselves at a fixed rate a non-zero entropy should not be permitted.

Our aim in this paper is twofold. In the first place we produce a simple model of irreversible dynamics and show that the statistical description in terms of partitions of unity can be used without problems. A second aim is to understand in a simple example which partitions can be allowed for. In particular, we will show that partitions that differ only 'infinitesimally' from the trivial partition form a reasonable class. More precisely, we will consider partitions generated by elements that are, up to compact perturbations, multiples of the identity. In fact, we will only work with a subset of the compact operators, namely the union of the von Neumann–Schatten classes $\mathcal{L}_p$ $(p \geqslant 1)$.

Section 5 contains the more technical lemmas needed for later calculations, in particular on approximations of compact operators by finite rank operators that are sufficiently good to control the entropies they generate. The main results are to be found in section 6. In a first lemma, we prove that $\log 2$ is an upper bound for the dynamical entropy of Θ with respect to the invariant state. A slight modification of this proof shows that the trivial dynamics, computed with compact partitions, produces zero entropy. In a second lemma, we put forward a partition of size 2 which reaches the entropy bound $\log 2$, leading to the conclusion that the entropy of our dynamical system equals $\log 2$.

2. Preliminaries

We will start this section by reminding ourselves of the construction of the dynamical entropy for a discrete dynamical system [3] and stating a continuity property.

Let $\mathcal{M}$ be a von Neumann algebra of operators acting on a Hilbert space $\mathcal{H}$ and let Ω be a normalized cyclic vector for $\mathcal{M}$, defining a state ω on $\mathcal{M}$. The single time-step evolution is given by an automorphism Θ (later on we will weaken this condition to a homomorphism) implemented by a unitary operator U such that $U\Omega = \Omega$ and $U\mathcal{M}U^* = \mathcal{M}$. In case a dynamical system is given in terms of a C^*-algebra one can make the Gelfand–Naimark–Segal (GNS) construction to obtain the von Neumann algebra picture.

An operational partition of unity of size k is a k-tuple $\mathcal{X}$ of elements $x_i \in \mathcal{M}$ satisfying

$$\sum_{i=0}^{k-1} x_i^* x_i = \mathbb{1}.$$

A partition $\mathcal{X} = (x_0, \dots, x_{k-1})$ evolves in time according to

$$\Theta(\mathcal{X}) := (\Theta(x_0), \dots, \Theta(x_{k-1})).$$

It can be composed with another partition $\mathcal{Y} = (y_0, \dots, y_{\ell-1})$ to yield

$$\mathcal{X} \circ \mathcal{Y} := (x_0 y_0, x_0 y_1, \dots, x_{k-1} y_{\ell-1})$$

which is of size $k\ell$.

To any partition $\mathcal{X}$ of size k we associate a $(k \times k)$ density matrix $\rho[\mathcal{X}]$ with (i, j) matrix element $\langle x_j \Omega, x_i \Omega \rangle$. The entropy $\mathsf{H}_{(\omega)}[\mathcal{X}]$ of the partition $\mathcal{X}$ is then

$$\mathsf{H}_{(\omega)}[\mathcal{X}] := \mathsf{S}(\rho[\mathcal{X}]) = \mathsf{S}\left(\sum_{i=0}^{k-1} |x_i \Omega\rangle \langle x_i \Omega| \right)$$

where the von Neumann entropy $\mathsf{S}(\rho)$ of a density matrix ρ is computed as $\mathrm{Tr}\, \eta(\rho)$ with $\eta(0) = 0$ and $\eta(x) = -x \log x$ for $0 < x \leqslant 1$. The equality of the two von Neumann entropies is a consequence of the fact that both density matrices have, up to multiplicities of zero, identical spectra [4]. To see this, one has to consider the vector $\Psi_{\mathcal{X}} = \sum_i e_i \otimes x_i \Omega$, $(e_0, \dots, e_{k-1})$ being a fixed orthonormal basis of $\mathbb{C}^k$. This vector is normalized and cyclic for $\mathcal{M}_k \otimes \mathcal{M}$. The restrictions of the pure vector state $|\Psi_{\mathcal{X}}\rangle\langle\Psi_{\mathcal{X}}|$ to $\mathcal{M}_k$ and $\mathcal{M}$, respectively, are exactly $\rho[\mathcal{X}]$ and $\sum_i |x_i \Omega\rangle\langle x_i \Omega|$.

By composing the partition $\mathcal{X}$ with its subsequent time evolutions we can construct larger and larger density matrices $\rho[\Theta^{n-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]$ on $\mathcal{M}_k^{\otimes[0,n-1]}$. These are right-compatible for different n in the sense that the partial trace over the last tensor factor, corresponding to time $(n-1)$, yields the density matrix up to time $(n-2)$. Therefore, these matrices define a state $\omega_{\mathcal{X}}$ on $\mathcal{M}_k^{\otimes\mathbb{N}}$. The dynamical entropy $\mathsf{H}_{(\Theta,\omega)}[\mathcal{X}]$ of the partition $\mathcal{X}$ is then the mean entropy density of $\omega_{\mathcal{X}}$, i.e.

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] = \limsup_n \frac{1}{n} \mathsf{H}_{(\omega)}[\Theta^{n-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}].$$

Consider a unital $*$-subalgebra $\mathcal{A}$ of $\mathcal{M}$ which is globally invariant under Θ. The dynamical entropy $\mathsf{h}_{(\Theta,\omega,\mathcal{A})}$ is obtained by taking the supremum of the dynamical entropy over all finite partitions in $\mathcal{A}$

$$\mathsf{h}_{(\Theta,\omega,\mathcal{A})} = \sup_{\mathcal{X}\subset\mathcal{A}} \mathsf{h}_{(\Theta,\omega)}[\mathcal{X}]. \tag{5}$$

The state $\omega_{\mathcal{X}}$ on $\mathcal{M}_k^{\otimes\mathbb{N}}$ can be expressed more explicitly by means of maps $\mathbb{E}_A$, where for $A = [a_{ij}]_{i,j=0,\ldots,k-1} \in \mathcal{M}_k$, $\mathbb{E}_A$ is given by [5]

$$\mathbb{E}_A : \mathcal{M} \to \mathcal{M} : x \mapsto \sum_{i,j} a_{ij} x_i^* \Theta(x) x_j.$$

A straightforward calculation yields $\omega_{\mathcal{X}}$ on elementary tensors

$$\omega_{\mathcal{X}}(A_0 \otimes \cdots \otimes A_{n-1}) = \omega((\mathbb{E}_{A_0} \circ \cdots \circ \mathbb{E}_{A_{n-1}})(\mathbb{1})).$$

It is this formula that we will use for the calculation of the dynamical entropy of a partition of unity. We first state a continuity property of the entropy $\mathsf{H}_{(\omega)}[\mathcal{X}]$ of a partition.

Lemma 1. Consider two families $\mathcal{X}^{(\alpha)} = (x_0^{(\alpha)}, \ldots, x_{k-1}^{(\alpha)})$ and $\mathcal{Y}^{(\alpha)} = (y_0^{(\alpha)}, \ldots, y_{k-1}^{(\alpha)})$ of partitions, $\alpha = 0, \ldots, n-1$, such that

$$\|x_i^{(\alpha)} - y_i^{(\alpha)}\| < \epsilon_\alpha \ (i = 0, \ldots, k-1) \qquad 2k \sum_{\alpha=0}^{n-1} \epsilon_\alpha < \frac{1}{3}.$$

Then

$$\left| \frac{1}{n} \mathsf{H}_{(\omega)}[\mathcal{X}^{(n-1)} \circ \cdots \circ \mathcal{X}^{(0)}] - \frac{1}{n} \mathsf{H}_{(\omega)}[\mathcal{Y}^{(n-1)} \circ \cdots \circ \mathcal{Y}^{(0)}] \right| \leqslant \left(2k \sum_{\alpha=0}^{n-1} \epsilon_\alpha\right) \log(2k) + \frac{1}{n} \eta\left(2k \sum_{\alpha=0}^{n-1} \epsilon_\alpha\right)$$

for any state ω.

For the proof we refer to [5].

The set of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is the norm closure of the ideal of finite rank operators or equivalently the set of operators that map uniformly bounded subsets of $\mathcal{H}$ into pre-compact sets [6]. Every compact operator A has an essentially unique, norm convergent expansion

$$A = \sum_{n\geqslant1} \mu_n |\xi_n\rangle\langle\phi_n|$$

where each $\mu_n > 0$, $\mu_1 \geqslant \mu_2 \geqslant \cdots$ and $\{\xi_n\}$ and $\{\phi_n\}$ are orthonormal sets. μ_n are the non-zero eigenvalues of $|A| = U^*A$, ϕ_n are the corresponding eigenvectors and $\xi_n = U\phi_n$. The possibility of degenerate eigenvalues of $|A|$ causes the lack of uniqueness.

For any $p \geqslant 1$ the Schatten class $\mathcal{L}_p$ is defined as

$$\mathcal{L}_p = \left\{ A \in \mathcal{K}(\mathcal{H}) \mid \|A\|_p = \left(\sum_{n \geqslant 1} \mu_n^p \right)^{1/p} < \infty \right\}.$$

For $p = 1$ we get the trace class operators and $p = 2$ corresponds to the Hilbert–Schmidt operators. Since $(\mu_n)_n$ is a decreasing sequence we can write

$$N \mu_N^p \leqslant \sum_{n=1}^{N} \mu_n^p < \|A\|_p^p$$

such that $\mu_N \leqslant \|A\|_p / N^{1/p}$. This allows us to formulate the following lemma.

Lemma 2. For any $A \in \mathcal{L}_p$ and any $\epsilon > 0$ there exists an operator A_N of finite rank $N - 1$ such that $\|A - A_N\| \leqslant \epsilon$ and

$$N = \left[\left(\frac{\|A\|_p}{\epsilon} \right)^p \right]$$

where $[x]$ denotes the smallest integer larger than x.

Proof. Choose $A \in \mathcal{L}_p$, $\epsilon > 0$ and set

$$N = \left[\left(\frac{\|A\|_p}{\epsilon} \right)^p \right] \qquad A_N = \sum_{n=1}^{N-1} \mu_n |\xi_n\rangle\langle\phi_n|.$$

Then

$$\|A - A_N\| = \left\| \sum_{n \geqslant N} \mu_n |\xi_n\rangle\langle\phi_n| \right\| = \mu_N \leqslant \frac{\|A\|_p}{N^{1/p}} \leqslant \epsilon. \qquad \square$$

3. Construction of the dynamics

We will look for a unity preserving $*$-homomorphism Θ of $\mathcal{B}(\mathcal{H})$ which satisfies equations (2) and (3) in the introduction and thus also equation (4). Consider therefore two operators $u_0, u_1 \in \mathcal{B}(\mathcal{H})$ which satisfy the Cuntz relations [7]

$$u_0 u_0^* + u_1 u_1^* = \mathbb{1} \qquad u_0^* u_0 = u_1^* u_1 = \mathbb{1}.$$

Equivalently, the map

$$u : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} : u = (u_0 \ u_1)$$

is unitary with adjoint

$$u^* : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} : u^* = \begin{pmatrix} u_0^* \\ u_1^* \end{pmatrix}.$$

The dynamics Θ on $\mathcal{B}(\mathcal{H})$ is then given as

$$\Theta(A) = u \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} u^* = (u_0 \ u_1) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} u_0^* \\ u_1^* \end{pmatrix}.$$

In fact, the previous formula is precisely the Stinespring decomposition of the $*$-homomorphism Θ of $\mathcal{B}(\mathcal{H})$.

We will show that (2) and (3) hold iff

$$(u_j \varphi)(x) = \exp(2\pi \mathrm{i} l_j x) \varphi(\theta(x)) \qquad (j = 0, 1)$$

where $l_0, l_1 \in \mathbb{Z}$ have different parity.

Writing (2) more explicitly as

$$u_0 M_f u_0^* + u_1 M_f u_1^* = M_{f\circ\theta}$$

and multiplying by u_0 on the right-hand side we get

$$u_0 M_f = M_{f\circ\theta} u_0.$$

Let us denote the constant function with value 1 by $\mathbf{1}$. Applying the previous line to $\mathbf{1}$ and evaluating this at a point $x \in [0, 1]$ we see that

$$(u_0 f)(x) = f(\theta(x))[u_0(\mathbf{1})](x).$$

Let $w \in \mathcal{B}(\mathcal{H})$ be the operator $(w\varphi)(x) := \varphi(\theta(x))$. A small computation shows that w is an isometry: $w^* w = \mathbb{1}$. We use this to write $f = w^* g$ with $wf = g$. This implies

$$(u_0 w^* g)(x) = [u_0(\mathbf{1})](x)(wf)(x) = [u_0(\mathbf{1})](x)g(x).$$

From this, we can conclude that $u_0(\mathbf{1}) \in \mathcal{L}^\infty([0, 1])$ and that $u_0 w^* = M_{u_0(\mathbf{1})}$. A similar argument can be given for u_1 leading to the existence of two essentially bounded functions f_0 and f_1 such that

$$u_j = M_{f_j} w \qquad (j = 0, 1).$$

The fact that Θ is a unity preserving homomorphism will impose some conditions on f_0 and f_1.

It is sufficient to check the validity of identity (3) for the multiplication operators M_f and the shifts U_{x_0} because the multiplication operators and the shift generate a strongly dense subalgebra of $\mathcal{B}(\mathcal{H})$ and we consider here strongly continuous homomorphisms. The relation

$$(\tau_{x_0} \circ \Theta)(M_f) = (\Theta \circ \tau_{2x_0})(M_f)$$

is a direct consequence of (1) applied to f. Using the definition of τ_{x_0} we see that

$$(\tau_{x_0} \circ \Theta)(U_{x_1}) = (\Theta \circ \tau_{2x_0})(U_{x_1})$$

is equivalent to

$$U_{x_0}\Theta(U_{x_1}) = \Theta(U_{x_1})U_{x_0}$$

for all x_0, x_1. The spectral decomposition of U_{x_0} reads

$$U_{x_0} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi k x_0} |\psi^k\rangle\langle\psi^k|$$

with $\psi^k(x) = \exp(\mathrm{i}2\pi k x)$ and

$$\begin{aligned}
\Theta(U_{x_1}) = \sum_k \mathrm{e}^{\mathrm{i}2\pi k x_1}\Theta(|\psi^k\rangle\langle\psi^k)
&= \sum_k \mathrm{e}^{\mathrm{i}2\pi k x_1}(|u_0\psi^k\rangle\langle u_0\psi^k| + |u_1\psi^k\rangle\langle u_1\psi^k|) \\
&= \sum_k \mathrm{e}^{\mathrm{i}2\pi k x_1}(|M_{f_0}\psi^{2k}\rangle\langle M_{f_0}\psi^{2k}| + |M_{f_1}\psi^{2k}\rangle\langle M_{f_1}\psi^{2k}|)
\end{aligned}$$

which is the spectral decomposition of $\Theta(U_{x_1})$. If we want U_{x_0} to commute with $\Theta(U_{x_1})$ we see that we have to impose $f_0 = \psi^{l_0}$ and $f_1 = \psi^{l_1}$. The conditions on f_0 and f_1 mentioned before are satisfied iff the parity of l_0 and l_1 are different.

Using the explicit formula

$$(u_j^*\varphi)(x) = \frac{1}{2}\exp(-\mathrm{i}\pi l_j x)\left(\varphi\left(\frac{x}{2}\right) + (-1)^{l_j}\varphi\left(\frac{x+1}{2}\right)\right) \qquad (j = 0, 1)$$

one easily verifies that relations (2) and (3) are satisfied if Θ is of the derived form.

4. Ergodic properties

Consider the inner product

$$(\rho, A) \mapsto \langle \rho, A\rangle := \operatorname{Tr} \rho A$$

between the trace class operators $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. The map Θ has a pre-adjoint Θ_* which, from $\langle \Theta_*(\rho), A\rangle = \langle \rho, \Theta(A)\rangle$, is explicitly given by

$$\Theta_*(\rho) = u_0^* \rho u_0 + u_1^* \rho u_1 \qquad \rho \in \mathcal{L}^1(\mathcal{H}).$$

We will look for density matrices that are invariant under Θ_* or rather for density matrices that describe the behaviour of normal states for large times. This will lead to a particular choice of the integers l_0 and l_1 in the definition of u_0 and u_1 in order to arrive at a dynamical system with optimal ergodic and mixing properties.

We will exclude the case where $|l_0 - l_1| > 1$ in order to avoid periodic behaviour. Suppose indeed that $l_0 < m < l_1$ with $m \in \mathbb{Z}$ and that l_0 is even (and hence l_1 odd). Using

$$u_0^* \psi^k = \begin{cases} \psi^{(-l_0+k)/2} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases} \qquad u_1^* \psi^k = \begin{cases} \psi^{(-l_1+k)/2} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}$$

one easily sees that $\Theta_*(|\psi^m\rangle\langle\psi^m|) = |\psi^r\rangle\langle\psi^r|$ with $l_0 < r < l_1$ and $r \neq m$. This means that subsequent applications of Θ_* will transform $|\psi^m\rangle\langle\psi^m|$ into itself via at least one intermediate $|\psi^r\rangle\langle\psi^r|$ ($l_0 < r < l_1$) and that we obtain a periodic behaviour. From now on we restrict our attention to $l_1 = l_0 + 1$.

Let us introduce the two projection operators

$$P_- = \sum_{k \leqslant -l_0 - 1} |\psi^k\rangle\langle\psi^k| \qquad P_+ = \sum_{k \geqslant -l_0} |\psi^k\rangle\langle\psi^k|$$

such that $P_- + P_+ = \mathbb{1}$. It takes a straightforward calculation to see that $u_\epsilon^* \psi^k \in P_\pm \mathcal{H}$ as soon as $\psi^k \in P_\pm \mathcal{H}$ ($\epsilon = 0, 1$), meaning that $P_\pm \Theta(A) P_\pm = \Theta(A)$ as soon as $P_\pm A P_\pm = A$. This allows us to consider only the subalgebra $\mathcal{A}$ of those operators for which $P_+ A P_+ = A$, or equivalently $\mathcal{B}(P_+\mathcal{H})$. The benefit of this will be the existence now of a unique invariant normal state $|\psi^{-l_0}\rangle\langle\psi^{-l_0}|$ (see theorem 1), whereas there are two invariant states $|\psi^{-l_0}\rangle\langle\psi^{-l_0}|$ and $|\psi^{-l_0-1}\rangle\langle\psi^{-l_0-1}|$ on $\mathcal{B}(\mathcal{H})$.

From now on we choose $l_0 = 0$ and hence $l_1 = 1$. This particular choice is quite convenient as it agrees well with the binary expansion of a natural number (see lemma 3). All subsequent results will be independent of the value of l_0. We have now arrived at the following model: consider in $\ell^2(\mathbb{N})$ the canonical orthonormal basis $\{\psi^0, \psi^1, \ldots,\}$ of sequences $\psi^j = (\delta_{jn})_{n\in\mathbb{N}}$ and the isometries u_0 and u_1 defined by

$$u_0^* \psi^{2k} = \psi^k \qquad u_0^* \psi^{2k+1} = 0 \qquad u_1^* \psi^{2k} = 0 \qquad u_1^* \psi^{2k+1} = \psi^k$$

for $k \in \mathbb{N}$. The single step time evolution on $\mathcal{A} = \mathcal{B}(\ell^2(\mathbb{N}))$ is given by

$$A \mapsto u_0 A u_0^* + u_1 A u_1^*.$$

Lemma 3. Introducing the notation $B(\epsilon_{n-1}, \ldots, \epsilon_0) = \sum_{i=0}^{n-1} \epsilon_i 2^i$ we have

$$u_{\epsilon_{n-1}}^* \ldots u_{\epsilon_0}^* \psi^{p+k2^n} = \begin{cases} \psi^k & \text{if } p = B(\epsilon_{n-1}, \ldots, \epsilon_0) \\ 0 & \text{otherwise} \end{cases}$$

with $\epsilon_{n-1}, \ldots, \epsilon_0 \in \{0, 1\}$, $0 \leqslant p \leqslant 2^n - 1$ and $k \in \mathbb{N}$.

Proof. We prove this by induction on n. The statement holds for $n = 1$. The induction step $n \to n+1$ can be seen as follows. We have to compute

$$u^*_{\epsilon_n} u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0} \psi^{p+k2^{n+1}}$$

with $0 \leqslant p \leqslant 2^{n+1} - 1$ and $k \in \mathbb{N}$. Therefore, we write $p = p' + k'2^n$ $(0 \leqslant p' \leqslant 2^n - 1)$. k' must be either 0 or 1 and

$$\begin{aligned} u^*_{\epsilon_n} u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0} \psi^{p+k2^{n+1}} &= u^*_{\epsilon_n} (u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0} \psi^{p'+k'2^n+k2^{n+1}}) \\ &= u^*_{\epsilon_n} (u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0} \psi^{p'+(k'+2k)2^n}) \\ &= \begin{cases} u^*_{\epsilon_n} \psi^{k'+2k} & \text{if } B(\epsilon_{n-1}, \dots, \epsilon_0) = p' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \psi^k & \text{if } k' = \epsilon_n \text{ and } B(\epsilon_{n-1}, \dots, \epsilon_0) = p' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The condition on the last but one line is equivalent to $B(\epsilon_n, \epsilon_{n-1}, \dots, \epsilon_0) = p$, which completes the proof. □

The next result shows that any state on $\mathcal{A}$ given by a density matrix relaxes to the unique invariant vector state determined by ψ^0.

Theorem 1. For any normal state ω_ρ on $\mathcal{A}$ given by $\omega_\rho(\cdot) = \mathrm{Tr}(\rho \cdot)$ we have

$$\lim_{n\to\infty} \|(\Theta_*)^n(\rho) - |\psi^0\rangle\langle\psi^0|\|_1 = 0.$$

Proof. It is sufficient to prove that

$$\lim_{n\to\infty} \|(\Theta_*)^n(|\varphi\rangle\langle\varphi|) - |\psi^0\rangle\langle\psi^0|\|_1 = 0$$

for any normalized vector $\varphi = \sum_{k\geqslant 0} \varphi_k \psi^k$ with $\varphi_k \in \mathbb{C}$.

The time evolved vector state can be written as

$$\begin{aligned} (\Theta_*)^n(|\varphi\rangle\langle\varphi|) &= \sum_{\epsilon_{n-1}\cdots\epsilon_0} |u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0}\varphi\rangle\langle u^*_{\epsilon_{n-1}} \dots u^*_{\epsilon_0}\varphi| \\ &= \sum_{p=0}^{2^n-1} \left| \sum_{p'=0}^{2^n-1} \sum_{k\geqslant 0} \varphi_{p'+k2^n} u^*_{(p)} \psi^{p'+k2^n} \right\rangle \left\langle \sum_{p'=0}^{2^n-1} \sum_{k\geqslant 0} \varphi_{p'+k2^n} u^*_{(p)} \psi^{p'+k2^n} \right| \end{aligned}$$

where we replace the summation over the ϵ's by the summation over their decimal expansion. We now use the previous lemma to write

$$\begin{aligned} (\Theta_*)^n(|\varphi\rangle\langle\varphi|) &= \sum_{p=0}^{2^n-1} \left| \sum_{k\geqslant 0} \varphi_{p+k2^n} \psi^k \right\rangle \left\langle \sum_{k\geqslant 0} \varphi_{p+k2^n} \psi^k \right| \\ &= \sum_{p=0}^{2^n-1} |\varphi_p \psi^0\rangle\langle\varphi_p \psi^0| + \sum_{p=0}^{2^n-1} |\varphi_p \psi^0\rangle \left\langle \sum_{k>0} \varphi_{p+k2^n} \psi^k \right| \\ &\quad + \sum_{p=0}^{2^n-1} \left| \sum_{k>0} \varphi_{p+k2^n} \psi^k \right\rangle \langle\varphi_p \psi^0| \\ &\quad + \sum_{p=0}^{2^n-1} \left| \sum_{k>0} \varphi_{p+k2^n} \psi^k \right\rangle \left\langle \sum_{k>0} \varphi_{p+k2^n} \psi^k \right|. \end{aligned}$$

It is clear that the first term will converge to $|\psi^0\rangle\langle\psi^0|$ in the trace norm. The trace norm of the remaining terms will vanish as n tends to infinity. We show this for the second term. The other terms can be treated in a similar way

$$\left\| \sum_{p=0}^{2^n-1} |\varphi_p \psi^0\rangle \left\langle \sum_{k>0} \varphi_{p+k2^n} \psi^k \right| \right\|_1 \leqslant \sum_{p=0}^{2^n-1} \left\| |\varphi_p \psi^0\rangle \left\langle \sum_{k>0} \varphi_{p+k2^n} \psi^k \right| \right\|_1$$

$$\leqslant \sum_{p=0}^{2^n-1} |\varphi_p| \left(\sum_{k>0} |\varphi_{p+k2^n}|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{p=0}^{2^n-1} |\varphi_p|^2 \right)^{\frac{1}{2}} \left(\sum_{p=0}^{2^n-1} \sum_{k>0} |\varphi_{p+k2^n}|^2 \right)^{\frac{1}{2}}.$$

The first factor will converge to $\|\varphi\|^2 = 1$ while the second factor equals

$$\left(\sum_{k \geqslant 2^n} |\varphi_k|^2 \right)^{\frac{1}{2}}$$

which converges to zero. □

Theorem 2. The spectrum of Θ with respect to the algebra $\mathcal{A}$ consists of the closed unit disc.

Proof. Let μ be any complex number in the open unit disc. We show that there exists a trace class operator ρ_μ such that $\Theta_*(\rho_\mu) = \mu\rho_\mu$. We propose

$$\rho_\mu = \sum_{i \geqslant 0} \lambda_i |\psi^i\rangle\langle\psi^i|$$

with $\lambda_i \in \mathbb{C}$. If ρ_μ is to be an eigenvector of Θ_*, the λ_i have to satisfy

$$\mu\lambda_i = \lambda_{2i} + \lambda_{2i+1}. \tag{6}$$

This requirement is fulfilled by the following choice of λ_i: set $\lambda_0 = 1$ and for $k \geqslant 0$ set

$$\lambda_{2^k} = \lambda_{2^k+1} = \cdots = \lambda_{2^{k+1}-1} = \frac{\mu^k(\mu - 1)}{2^k}.$$

This makes ρ_μ into a trace class operator

$$\begin{aligned} \sum_{i \geqslant 0} |\lambda_i| &= 1 + \sum_{k \geqslant 0} \sum_{m=0}^{2^k-1} |\lambda_{2^k+m}| \\ &= 1 + |\mu - 1| \sum_{k \geqslant 0} \sum_{m=0}^{2^k-1} \frac{|\mu|^k}{2^k} \\ &= 1 + |\mu - 1| \sum_{k \geqslant 0} |\mu|^k \end{aligned}$$

and this infinite sum is converging since $|\mu| < 1$. Furthermore, it is clear that $\mu\lambda_0 = \lambda_0 + \lambda_1$ and by writing $i \geqslant 1$ as $i = 2^k + m$ with $k \geqslant 0$ and $0 \leqslant m \leqslant 2^k - 1$ one easily sees that condition (6) is met for all i.

As Θ is the adjoint of Θ_*, its spectrum contains at least the closed unit disc. On the other hand it is contained in the same disc because Θ has norm 1. Therefore the statement follows. □

Theorem 3. The only eigenvalues of Θ are 0 and 1. Up to scalar multiples, the only eigenvector of Θ corresponding the eigenvalue 1 is the unit operator.

Proof. Let μ be in the spectrum and suppose that A is a corresponding eigenvector. Choose integers $k, l \geqslant 0$ and take a positive integer such that $k, l \leqslant 2^n - 1$. From $\Theta^n(A) = \mu^n A$ we have

$$\begin{aligned}\mu^n \langle \psi^k, A\psi^l \rangle &= \langle \psi^k, \Theta^n(A)\psi^l \rangle \\ &= \sum_{p=0}^{2^n-1} \langle u_{(p)}^* \psi^k, A u_{(p)}^* \psi^l \rangle \\ &= \begin{cases} 0 & \text{if } k \neq l \\ \langle \psi^0, A\psi^0 \rangle & \text{if } k = l. \end{cases}\end{aligned}$$

If $\mu \neq 0$ and $\mu \neq 1$, we can conclude that $\langle \psi^k, A\psi^l \rangle = 0$ for all $k, l \geqslant 0$ and so $A = 0$. Therefore, only 0 and 1 can be eigenvalues.

For $\mu = 1$ we see that $\langle \psi^k, A\psi^l \rangle = 0$ if $k \neq l$ and $\langle \psi^k, A\psi^k \rangle = \langle \psi^0, A\psi^0 \rangle$ for all $k \geqslant 0$ and hence $A \in \mathbb{C}\mathbb{1}$.

To end the proof we mention that the eigenvectors corresponding to the eigenvalue 0 are those operators for which $\langle \psi^k, A\psi^l \rangle = 0$ if k and l have the same parity. □

Lemma 4. The pure state $\omega_0 : A \mapsto \langle \psi^0, A\psi^0 \rangle$ is mixing under Θ, i.e.

$$\lim_{n\to\infty} \omega_0(\Theta^n(A)B) = \omega_0(A)\omega_0(B)$$

for any two operators $A, B \in \mathcal{A}$.

Proof. For any two operators A and B in $\mathcal{A}$, we have

$$\begin{aligned}\omega_0(\Theta^n(A)B) &= \sum_{p=0}^{2^n-1} \langle \psi^0, u_{(p)} A u_{(p)}^* B\psi^0 \rangle \\ &= \langle \psi^0, A u_0^{*n} B\psi^0 \rangle \\ &= \sum_{k\geqslant 0} \langle \psi^0, A u_0^{*n} \psi^k \rangle \langle \psi^k, B\psi^0 \rangle \\ &= \sum_{k\geqslant 0} \langle \psi^0, A\psi^k \rangle \langle \psi^{2^n k}, B\psi^0 \rangle\end{aligned}$$

so, separating out the term $k = 0$, we see

$$\begin{aligned}|\omega_0(\Theta^n(A)B) - \omega_0(A)\omega_0(B)| &\leqslant \sum_{k\geqslant 1} |\langle \psi^0, A\psi^k \rangle||\langle \psi^{2^n k}, B\psi^0 \rangle| \\ &\leqslant \left(\sum_{k\geqslant 1} |\langle \psi^0, A\psi^k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k\geqslant 1} |\langle \psi^{2^n k}, B\psi^0 \rangle|^2 \right)^{\frac{1}{2}} \\ &\leqslant \|A^*\psi^0\| \left(\sum_{k\geqslant 2^n} |\langle \psi^k, B\psi^0 \rangle|^2 \right)^{\frac{1}{2}}\end{aligned}$$

which tends to zero as n goes to infinity. □

5. Technical lemmas

Lemma 5. An operator $x = \alpha\mathbb{1} + K$ ($\alpha \in \mathbb{C}$, $K \in \mathcal{L}_p$) on a Hilbert space $\mathcal{H}$ can be written in the form $x = U|x|$ where U is unitary and $U, |x| \in \mathbb{C}\mathbb{1} + \mathcal{L}_p$.

Proof. Take as a starting point the polar decomposition of $x = \hat{U}|x|$ where $\hat{U}$ is defined as

$$\begin{aligned} &\hat{U}|x|\varphi = x\varphi \\ &\hat{U}\varphi = 0 \qquad \text{if } \varphi \in \mathrm{Ker}(|x|) = \mathrm{Ker}\,(x). \end{aligned}$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be Fredholm if its range is closed and both $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)^\perp$ are finite-dimensional. For Fredholm operators, the index is defined by $\mathrm{Ind}(A) := \dim \mathrm{Ker}(A) - \dim \mathrm{Ran}(A)^\perp$. If A is Fredholm and C is compact, then $A + C$ is Fredholm and $\mathrm{Ind}(A) = \mathrm{Ind}(A + C)$ [8].

Because $\mathrm{Ind}(\mathbb{1}) = 0$, we have $\mathrm{Ind}(x) = 0$ and therefore

$$\dim \mathrm{Ker}(x) = \dim \mathrm{Ran}(x)^\perp < \infty.$$

This admits the definition of a finite rank isomorphism

$$\Phi : \mathrm{Ker}(x) \to \mathrm{Ran}(x)^\perp$$

which can be extended to $\mathcal{H}$ by putting $\mathrm{Ker}(\Phi) = \mathrm{Ran}(|x|)$.

If we define $U = \hat{U} + \Phi$ we can immediately verify that $UU^* = \mathbb{1}$

$$\begin{aligned} UU^* &= \hat{U}\hat{U}^* + \Phi\Phi^* + \hat{U}\Phi^* + \Phi\hat{U}^* \\ &= \mathrm{P}_{\mathrm{Ran}(x)} + \mathrm{P}_{\mathrm{Ran}(x)^\perp} = \mathbb{1} \end{aligned}$$

as $\hat{U}\Phi^* = \Phi\hat{U}^* = 0$. Along the same lines

$$\begin{aligned} U^*U &= \hat{U}^*\hat{U} + \Phi^*\Phi + \Phi^*\hat{U} + \hat{U}^*\Phi \\ &= \mathrm{P}_{\mathrm{Ran}(x^*)} + \mathrm{P}_{\mathrm{Ker}(x)} = \mathbb{1} \end{aligned}$$

as $\Phi^*\hat{U} = \hat{U}^*\Phi = 0$.

It remains to prove that U and $|x|$ belong to $\mathbb{C}\mathbb{1} + \mathcal{L}_p$. From

$$|x|^2 = x^*x = |\alpha|^2\mathbb{1} + \alpha K^* + \bar{\alpha}K + K^*K$$

we see that $(|x| + |\alpha|\mathbb{1})(|x| - |\alpha|\mathbb{1}) \in \mathcal{L}_p$. Because $|x|$ is positive, $|x| + |\alpha|\mathbb{1}$ is invertible and this implies that $|x| \in |\alpha|\mathbb{1} + \mathcal{L}_p$ since $\mathcal{L}_p$ is an ideal. Furthermore, $x = U|x| = U(|\alpha|\mathbb{1} + L)$ with $L \in \mathcal{L}_p$ so $|\alpha|U \in \alpha\mathbb{1} + \mathcal{L}_p$ or finally $U \in (\alpha/|\alpha|)\mathbb{1} + \mathcal{L}_p$. □

Lemma 6. For any partition of unity $\mathcal{X} = (x_0, \ldots, x_{k-1}) \subset \mathcal{B}(\mathcal{H})$ in elements of the form

$$x_i = \alpha_i\mathbb{1} + K_i \qquad \alpha_i \in \mathbb{C}, \quad K_i \in \mathcal{L}_p$$

there is a constant C such that we can construct for every $\epsilon > 0$ a partition $\mathcal{Y}$ with $\|x_i - y_i\| < \epsilon$ $(i = 0, \ldots, k-1)$. Furthermore, $\mathcal{Y}$ is of the form

$$y_i = \beta_i\mathbb{1} + \tilde{K}_i \qquad \beta_i \in \mathbb{C}, \quad \tilde{K}_i \text{ finite rank}$$

with $\tilde{K}_i = P_{\mathrm{fin}}\tilde{K}_i P_{\mathrm{fin}}$ where P_{fin} is a projection of dimension

$$N = 2k\left[\left(\frac{C}{\epsilon^2}\right)^p\right]$$

at most.

Proof. Choose $\epsilon > 0$ and let $0 < \gamma < 1$ be a real number, close to 1, which we will specify later on and will depend on ϵ. Define k operators $z_i = \gamma x_i$. For this k-tuple we see that $\|x_i - z_i\| \leqslant 1 - \gamma$ and $\|\sum_{i=0}^{k-2} z_i^* z_i\| \leqslant \gamma^2$.

The first $(k-1)$ elements of $\mathcal{Y}$ will be defined by cutting off the compact operators, appearing in z_i, to finite rank operators such that $\sum_{i=0}^{k-2} y_i^* y_i \leqslant \mathbb{1}$. One way to achieve this, is to require that

$$\left\| \sum_{i=0}^{k-2} z_i^* z_i - \sum_{i=0}^{k-2} y_i^* y_i \right\| < 1 - \gamma^2.$$

This requirement can be met by asking that $y_{i=0,\dots,k-2}$ should satisfy

$$\|z_i - y_i\| < \frac{1-\gamma^2}{3(k-1)}$$

since one then has

$$\begin{aligned}
\left\| \sum_{i=0}^{k-2} z_i^* z_i - \sum_{i=0}^{k-2} y_i^* y_i \right\| &\leqslant \sum_{i=0}^{k-2} \|z_i^*\|\|z_i - y_i\| + \|z_i^* - y_i^*\|\|y_i\| \\
&\leqslant \sum_{i=0}^{k-2} \gamma\|z_i - y_i\| + \|z_i - y_i\|(\|z_i - y_i\| + \gamma) \\
&\leqslant \sum_{i=0}^{k-2} 2\|z_i - y_i\| + \|z_i - y_i\|^2 \\
&\leqslant 3(k-1) \max_{i=0,\dots,k-2} \|z_i - y_i\|.
\end{aligned}$$

Let us fix a collection y_i $(i = 0, \dots, k-2)$ making sure that

$$1 - \gamma + \frac{1-\gamma^2}{3(k-1)} < \epsilon.$$

To construct y_{k-1}, we use lemma 5 to write

$$\begin{aligned}
x_{k-1} &= U|x_{k-1}| \\
&= U\sqrt{\mathbb{1} - \sum_{i=0}^{k-2} x_i^* x_i}
\end{aligned}$$

where $U = \mathrm{e}^{\mathrm{i}\theta_0}\mathbb{1} + L$ $(L \in \mathcal{L}_p)$ is a unitary operator. Because of the unitarity of U, the structure of L will be

$$\begin{aligned}
L &= \sum_{n\geqslant 1} (\mathrm{e}^{\mathrm{i}\theta_n} - \mathrm{e}^{\mathrm{i}\theta_0})|\varphi_n\rangle\langle\varphi_n| \\
&= \sum_{n\geqslant 1} |\mathrm{e}^{\mathrm{i}\theta_n} - \mathrm{e}^{\mathrm{i}\theta_0}||\xi_n\rangle\langle\varphi_n|
\end{aligned}$$

which gives us the canonical decomposition of L provided that we arrange the φ_n in such a way that $|\exp(\mathrm{i}\theta_n) - \exp(\mathrm{i}\theta_0)|$ is a decreasing sequence, converging to 0.

Approximating L by a finite rank operator $\tilde{L}$, i.e. restricting the norm convergent sum to a finite number of terms, we get a new unitary $\tilde{U} = \exp(\mathrm{i}\theta_0)\mathbb{1} + \tilde{L}$. We now put

$$y_{k-1} := \tilde{U}\sqrt{\mathbb{1} - \sum_{i=0}^{k-2} y_i^* y_i}.$$

Note that $\mathcal{Y}$ is a partition of unity because $\tilde{U}^*\tilde{U} = \mathbb{1}$. The norm difference between z_{k-1} and y_{k-1} can be estimated as follows

$$\begin{aligned}\|y_{k-1} - z_{k-1}\| &\leqslant \left\|\tilde{U}\sqrt{\mathbb{1} - \sum_{i=0}^{k-2} y_i^* y_i} - U\sqrt{\gamma^2\mathbb{1} - \sum_{i=0}^{k-2} z_i^* z_i}\right\| \\ &\leqslant \|\tilde{U}\|\left\|\sqrt{\mathbb{1} - \sum_{i=0}^{k-2} y_i^* y_i} - \sqrt{\gamma^2\mathbb{1} - \sum_{i=0}^{k-2} z_i^* z_i}\right\| \\ &\quad + \|\tilde{U} - U\|\left\|\sqrt{\gamma^2\mathbb{1} - \sum_{i=0}^{k-2} z_i^* z_i}\right\|.\end{aligned}$$

Using lemma 9 and applying the triangle inequality we get

$$\begin{aligned}\|y_{k-1} - z_{k-1}\| &\leqslant \sqrt{6}\left\|(1-\gamma^2)\mathbb{1} + \sum_{i=0}^{k-2}(z_i^* z_i - y_i^* y_i)\right\|^{1/2} + \|\tilde{U} - U\| \\ &\leqslant \sqrt{12}\sqrt{1-\gamma^2} + \|\tilde{U} - U\|.\end{aligned}$$

If we now determine $\tilde{U}$ in such a way that $\|\tilde{U} - U\| < (1-\gamma^2)^{1/2}$ and if we put

$$\sqrt{1-\gamma^2} = \frac{\epsilon}{6}$$

then $\|x_i - y_i\| < \epsilon$ $(i = 0, \dots, k-1)$ since for $i = 0, \dots, k-2$

$$\begin{aligned}\|x_i - y_i\| &\leqslant \|x_i - z_i\| + \|z_i - y_i\| < (1-\gamma) + \frac{1-\gamma^2}{3(k-1)} \\ &< (1-\gamma^2) + (1-\gamma^2) < 2\sqrt{1-\gamma^2} = \frac{\epsilon}{3} < \epsilon\end{aligned}$$

and also

$$\begin{aligned}\|x_{k-1} - y_{k-1}\| &\leqslant \|x_{k-1} - z_{k-1}\| + \|z_{k-1} - y_{k-1}\| \leqslant (1-\gamma) + \sqrt{12}\sqrt{1-\gamma^2} + \sqrt{1-\gamma^2} \\ &< \sqrt{1-\gamma^2} + \sqrt{12}\sqrt{1-\gamma^2} + \sqrt{1-\gamma^2} < 6\sqrt{1-\gamma^2} < \epsilon.\end{aligned}$$

The question remains about the number of dimensions on which the $\tilde{K}_i$, appearing in y_i, live. The y_i for $i = 0, \dots, k-2$ were defined by demanding that the norm difference between the compact operator in z_i (which is in $\mathcal{L}_p$) and the finite rank in y_i be smaller than

$$\frac{1-\gamma^2}{3(k-1)} = \frac{\epsilon^2}{108(k-1)}$$

meaning that we can choose $\tilde{K}_i$ $(i = 0, \dots, k-2)$ to be of rank

$$\left[\left(\frac{108(k-1)C_p}{\epsilon^2}\right)^p\right]$$

where $C_p = \max\{\|K_0\|_p, \dots, \|K_{k-2}\|_p, \|L\|_p\}$. From this we can conclude that there is a finite-dimensional subspace $M \subset \mathcal{H}$ of dimension at most

$$2(k-1)\left[\left(\frac{108(k-1)C_p}{\epsilon^2}\right)^p\right]$$

such that $y_i = P_M y_i P_M + \beta_i P_{M^\perp}$ $(i = 0, \ldots, k-2)$ and $|y_{k-1}| = P_M|y_{k-1}|P_M + |\beta_{k-1}|P_{M^\perp}$. The last aspect to be considered is the rank of $\tilde{L}$. Since

$$\frac{\epsilon^2}{108(k-1)} < \frac{\epsilon^2}{36} < \frac{\epsilon}{6}$$

we can give a rough estimate that there exists a finite-dimensional subspace $P_{\text{fin}}\mathcal{H}$ of at most dimension

$$2k\left[\left(\frac{108(k-1)C_p}{\epsilon^2}\right)^p\right]$$

such that $\tilde{K}_i = P_{\text{fin}}\tilde{K}_i P_{\text{fin}}$. The proof is finished by putting $C = 108(k-1)C_p$. □

6. Dynamical entropy

Lemma 7. For any partition of unity $\mathcal{X} = (x_0, \ldots, x_{k-1})$ in $\mathcal{A}$ of the form

$$x_i = \alpha_i \mathbb{1} + K_i \qquad (\alpha_i \in \mathbb{C}, K_i \in \mathcal{L}_p)$$

we have:

$$\mathsf{h}_{(\Theta,\omega_0)}[\mathcal{X}] \leqslant \log 2.$$

Proof. Choose $\epsilon > 0$ and consider the decreasing sequence $\epsilon_\alpha := \epsilon/(\alpha+1)^2$. By taking ϵ sufficiently small we can make

$$2k\sum_{\alpha\geqslant 0}\epsilon_\alpha$$

arbitrarily small. Using lemma 6 we can find a sequence of partitions $\hat{\mathcal{Y}}^{(\alpha)} = (\hat{y}_0^{(\alpha)}, \ldots, \hat{y}_{k-1}^{(\alpha)})$ and a sequence of projections $\hat{P}_\alpha$ $(\alpha \geqslant 0)$ satisfying

$$\|x_i - \hat{y}_i^{(\alpha)}\| < \epsilon_\alpha \qquad (i = 0, \ldots, k-1)$$
$$\hat{y}_i^{(\alpha)} = \hat{P}_\alpha \hat{y}_i^{(\alpha)} \hat{P}_\alpha + \beta_i \hat{P}_\alpha^\perp \qquad (i = 0, \ldots, k-1)$$
$$\dim(\hat{P}_\alpha(\mathcal{H})) = \hat{N}_\alpha = 2k\left[\left(\frac{C(\alpha+1)^4}{\epsilon^2}\right)^p\right].$$

We now consider the sequence $\mathcal{X}^{(\alpha)} = \Theta^\alpha(\mathcal{X})$ and $\mathcal{Y}^{(\alpha)} = \Theta^\alpha(\hat{\mathcal{Y}}^{(\alpha)})$. Since Θ is not norm increasing

$$\|x_i^{(\alpha)} - y_i^{(\alpha)}\| < \epsilon_\alpha \qquad (i = 0, \ldots, k-1)$$

still holds. Furthermore, because

$$\Theta(|\xi\rangle\langle\chi|) = |u_0\xi\rangle\langle u_0\chi| + |u_1\xi\rangle\langle u_1\chi|$$

there exists a sequence of projections P_α such that

$$y_i^{(\alpha)} = P_\alpha y_i^{(\alpha)} P_\alpha + \beta_i P_\alpha^\perp \qquad (i = 0, \ldots, k-1)$$
$$\dim(P_\alpha(\mathcal{H})) = N_\alpha = 2^{\alpha+1}k\left[\left(\frac{C(\alpha+1)^4}{\epsilon^2}\right)^p\right].$$

From this we can conclude that the density matrix $\rho[\mathcal{Y}^{(n-1)} \circ \cdots \circ \mathcal{Y}^{(0)}]$ will be living on a subspace of dimension bounded by

$$1 + \sum_{\alpha=0}^{n-1} N_\alpha \leqslant 1 + k\,2^n\left(\frac{C}{\epsilon^2}\right)^p \sum_{\alpha=0}^{n-1}\alpha^{4p} \leqslant 1 + k\,2^n\left(\frac{C}{\epsilon^2}\right)^p n^{4p+1}$$

and hence

$$\lim_{n\to\infty}\frac{1}{n}\mathsf{H}_{(\omega_0)}[\mathcal{Y}^{(n-1)}\circ\cdots\circ\mathcal{Y}^{(0)}]\leqslant\lim_{n\to\infty}\frac{1}{n}\log\left(1+k\,2^n\left(\frac{C}{\epsilon^2}\right)^p n^{4p+1}\right)\leqslant\log 2.$$

Since we can make the right-hand side of the estimate of lemma 1 arbitrarily small, this finishes the proof. □

Lemma 8. $\mathcal{X}\mapsto\mathsf{h}_{(\Theta,\omega_0)}[\mathcal{X}]$ reaches its upper bound $\log 2$ on the partition $\mathcal{X}=(x_0,x_1)$ given by

$$x_k=\frac{1}{\sqrt{2}}(\mathbb{1}+(\mathrm{i}-1)|\xi_k\rangle\langle\xi_k|)$$
$$\xi_k=\frac{1}{\sqrt{2}}(\psi^0+(-1)^k\psi^1)\qquad k=0,1.$$

Proof. The lemma follows from the fact that the state $\omega_{\mathcal{X}}$ on $\mathcal{M}_2^{\otimes\mathbb{N}}$ is the normalized trace, i.e.

$$\begin{aligned}\omega_{\mathcal{X}}(A^{m-1}\otimes\cdots\otimes A^0)&=\langle\psi^0,\mathbb{E}_{A^{m-1}}\circ\cdots\circ\mathbb{E}_{A^0}(\mathbb{1})\psi^0\rangle\\&=\mathrm{tr}\,(A^{m-1}\otimes\cdots\otimes A^0).\end{aligned}$$

In order to prove this, we introduce the notation $B\equiv C(B,C\in\mathcal{B}(\mathcal{H}))$ iff $B-C$ is a linear combination of rank one operators of the form $|\psi^k\rangle\langle\psi^l|$ $(k,l\geqslant 0)$ with k or l different from 0. In that case $\omega_0(B)=\omega_0(C)$, where ω_0 is the vector state defined by ψ^0. So it is sufficient to prove that

$$\begin{aligned}(\mathbb{E}_{A^{m-1}}\circ\cdots\circ\mathbb{E}_{A^0})(\mathbb{1})\equiv&\left(\mathrm{tr}(A^{m-1}\otimes\cdots\otimes A^0)-\prod_{k=0}^{m-1}\frac{A_{00}^k+A_{11}^k+A_{01}^k+A_{10}^k}{2}\right)|\psi^0\rangle\langle\psi^0|\\&+\left(\prod_{k=0}^{m-1}\frac{(A_{00}^k+A_{11}^k+A_{01}^k+A_{10}^k)}{2}\right)\mathbb{1}\end{aligned}\tag{7}$$

since $\omega_0(\mathbb{1})=\omega_0(|\psi^0\rangle\langle\psi^0|)=1$. This can be done by induction

$$\begin{aligned}\mathbb{E}_A(\mathbb{1})=\sum_{i,j}A_{ij}x_j^*x_i&=A_{00}x_0^*x_0+A_{11}x_1^*x_1+A_{01}x_1^*x_0+A_{10}x_0^*x_1\\&=\frac{A_{00}+A_{11}}{2}\mathbb{1}+\frac{A_{01}}{2}(\mathbb{1}+(\mathrm{i}-1)|\xi_0\rangle\langle\xi_0|+(-\mathrm{i}-1)|\xi_1\rangle\langle\xi_1|)\\&\quad+\frac{A_{10}}{2}(\mathbb{1}+(\mathrm{i}-1)|\xi_1\rangle\langle\xi_1|+(-\mathrm{i}-1)|\xi_0\rangle\langle\xi_0|).\end{aligned}$$

Expanding $|\xi_k\rangle\langle\xi_k|$ and collecting the terms containing $\mathbb{1}$ and $|\psi^0\rangle\langle\psi^0|$, yields

$$\mathbb{E}_A(\mathbb{1})\equiv\left(\frac{A_{00}+A_{11}+A_{01}+A_{10}}{2}\right)\mathbb{1}-\left(\frac{A_{01}+A_{10}}{2}\right)|\psi^0\rangle\langle\psi^0|.$$

To prove the induction step from $(m-1)$ to m we need to check two points. The first one is

$$\mathbb{E}_A(|\psi^0\rangle\langle\psi^0|)\equiv\left(\frac{A_{00}+A_{11}}{2}\right)|\psi^0\rangle\langle\psi^0|$$

which can be seen by a calculation similar to that of $\mathbb{E}_A(\mathbb{1})$. The second point to notice is that $\mathbb{E}_A(B)\equiv\mathbb{E}_A(C)$ as soon as $B\equiv C$, which reflects the fact that $\mathbb{E}_A(|\psi^k\rangle\langle\psi^l|)$ (k or l different from 0) cannot be equal to $\mathbb{1}$ or $|\psi^0\rangle\langle\psi^0|$.

If we suppose that (7) is true for m, we have for $(m+1)$:

$$
\begin{aligned}
(\mathbb{E}_{A^m} \circ \cdots \circ \mathbb{E}_{A^0})(\mathbb{1}) &= \mathbb{E}_{A^m}((\mathbb{E}_{A^{m-1}} \circ \ldots \circ \mathbb{E}_{A^0})(\mathbb{1})) \\
&\equiv \left(\operatorname{tr}(A^{m-1} \otimes \cdots \otimes A^0) - \prod_{k=0}^{m-1} \frac{A_{00}^k + A_{11}^k + A_{01}^k + A_{10}^k}{2} \right) \frac{A_{00}^m + A_{11}^m}{2} |\psi^0\rangle\langle\psi^0| \\
&\quad + \left(\prod_{k=0}^{m-1} \frac{A_{00}^k + A_{11}^k + A_{01}^k + A_{10}^k}{2} \right) \frac{A_{00}^m + A_{11}^m + A_{01}^m + A_{10}^m}{2} \mathbb{1} \\
&\quad - \left(\prod_{k=0}^{m-1} \frac{A_{00}^k + A_{11}^k + A_{01}^k + A_{10}^k}{2} \right) \frac{A_{01}^m + A_{10}^m}{2} |\psi^0\rangle\langle\psi^0| \\
&\equiv \left(\operatorname{tr}(A^{m} \otimes \cdots \otimes A^0) - \prod_{k=0}^{m} \frac{A_{00}^k + A_{11}^k + A_{01}^k + A_{10}^k}{2} \right) |\psi^0\rangle\langle\psi^0| \\
&\quad + \left(\prod_{k=0}^{m} \frac{(A_{00}^k + A_{11}^k + A_{01}^k + A_{10}^k)}{2} \right) \mathbb{1}.
\end{aligned}
$$

Taking the expectation value ω_0 of expression, we obtain the desired result. □

From the last two lemmas we can conclude the following theorem.

Theorem 4. The dynamical entropy of Θ with respect to the state ω_0 is

$$
\mathsf{h}_{(\Theta,\omega_0)} = \log 2
$$

where the supremum in definition (5) of the dynamical entropy is taken over $\mathcal{L}_p$ partitions of unity.

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Appendix

Lemma 9. Let $0 \leqslant A, B \leqslant \mathbb{1}$ be two operators on a Hilbert space. Then

$$
\|A^{1/2} - B^{1/2}\| \leqslant \sqrt{6}\|A - B\|^{1/2}.
$$

Proof. Starting from the spectral decomposition

$$
A = \int_0^1 \lambda \, \mathrm{d}E_A(\lambda) \qquad B = \int_0^1 \lambda \, \mathrm{d}E_B(\lambda)
$$

we set

$$
\begin{aligned}
A_s &= \int_0^\gamma \lambda \, \mathrm{d}E_A(\lambda) \qquad & A_l &= \int_\gamma^1 \lambda \, \mathrm{d}E_A(\lambda) \\
B_s &= \int_0^\gamma \lambda \, \mathrm{d}E_B(\lambda) \qquad & B_l &= \int_\gamma^1 \lambda \, \mathrm{d}E_B(\lambda)
\end{aligned}
$$

with $0 < \gamma < 1$. We obtain the following inequalities

$$\begin{aligned}\|\sqrt{A} - \sqrt{B}\| &= \|\sqrt{A_l + A_s} - \sqrt{B_l + B_s}\| \\ &\leqslant \|\sqrt{A_l} - \sqrt{B_l}\| + \|\sqrt{A_s} - \sqrt{B_s}\| \\ &\leqslant \|\sqrt{A_l} - \sqrt{B_l}\| + 2\sqrt{\gamma}.\end{aligned}$$

Using the series expansion $\sqrt{1 - z} = 1 - \sum_{k\geqslant 1} a_k z^k$ and applying a telescoping argument

$$\begin{aligned}\|\sqrt{A_l} - \sqrt{B_l}\| &= \|\sqrt{\mathbb{1} - (\mathbb{1} - A_l)} - \sqrt{\mathbb{1} - (\mathbb{1} - B_l)}\| \\ &\leqslant \sum_k a_k \|(\mathbb{1} - A_l)^k - (\mathbb{1} - B_l)^k\| \\ &\leqslant \sum_k a_k k (1 - \gamma)^{k-1} \|A_l - B_l\| \\ &\leqslant \|A_l - B_l\| \frac{1}{2\sqrt{\gamma}} \\ &\leqslant (\|A_l - A\| + \|A - B\| + \|B - B_l\|) \frac{1}{2\sqrt{\gamma}} \\ &\leqslant (2\gamma + \|A - B\|) \frac{1}{2\sqrt{\gamma}}\end{aligned}$$

which results in the estimate

$$\|\sqrt{A} - \sqrt{B}\| \leqslant 3\sqrt{\gamma} + \frac{\|A - B\|}{2\sqrt{\gamma}}.$$

Putting $\gamma = \|A - B\|/6$, which optimizes the previous line, we obtain the result. □

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